

## CHERN-SIMONS SOLITONS IN QUANTUM POTENTIAL

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**Abstract**—*The self-dual Chern-Simons solitons under the influence of the quantum potential are considered. The single-valuedness condition for an arbitrary integer number  $N \geq 0$  of solitons leads to quantization of Chern-Simons coupling constant  $\kappa = m \frac{e^2}{g}$ , and the integer strength of quantum potential  $s = 1 - m^2$ . As we show, the Jackiw-Pi model corresponds to the first member ( $m = 1$ ) of our hierarchy of the Chern-Simons gauged nonlinear Schrödinger models, admitting self-dual solitons. New type of exponentially localized Chern-Simons solitons for the Bloch electrons near the hyperbolic energy band boundary are found.*

## 1. INTRODUCTION

The Chern-Simons theory attracted much attention recently as a gauge theory in 2+1 dimensions [1], describing non-local interaction between matter particles, affecting the phase of the wave function, and leading to the fractional statistics phenomena [2]. In this connection the Chern-Simons coupling constant plays the role of a statistical parameter and relates to the spin of particles. At the classical non-relativistic level, with the Chern-Simons coupling strength determined by overall strength of a quartic scalar potential, it leads to the existence of the self-dual Chern-Simons solitons [3]. They are static solutions with finite charge and flux of exactly solvable self-dual equations [4]. Unfortunately, the full dynamics of Chern-Simons solitons according to the nonlinear Schrödinger equation is still inflexible due to the lack of exact integrability, even without external forces, although it may be realized in the framework of the Davey-Stewartson-II equation [5].

In the present paper we show that the "external" force produced by the so-called "quantum potential"  $U(x) = (-\hbar^2/2m)\Delta|\psi|/|\psi|$ , leads to an additional statistical transmutation of Chern-Simons solitons. The quantum potential, introduced by L. de Broglie [6] and explored by D. Bohm [7] does not depend on the strength of the wave but only on its form, and therefore its effect could be large even at long distances. Then, it satisfies the homogeneity property [8], this is the reason why it appears in attempts of a nonlinear extension of the quantum mechanics [9,11-14]. An approach has been developed to the

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stochastic formulation of quantum mechanics, where the quantum fluctuations are represented by superimposing a classical trajectory and an additional random motion generated by quantum potential [10]. The nonlinear extension of the Schrödinger equation with the quantum potential non-linearity has been considered in connection with several problems [11]: a) in allowing formally the diffusion coefficient of the stochastic process in a stochastic quantization to differ  $\hbar/2m$ , related to the difference in the Plank constant [12] or the inertial mass [13], b) in corrections from quantum gravity [14]. As was shown by Sabatier, depending from intensity of the quantum potential it can be linearized in the form of the Schrödinger equation with rescaled potential or as the pair of time reversed diffusion equations [15], and in both cases does not admit soliton solutions. Recently we find that for self-consistent potential  $U = g|\psi|^2$  in 1+1 dimensions the theory has soliton solutions with rich resonance dynamics [16]. In the present paper we find exactly soluble case for 2+1 dimensional nonlinear Schrödinger (NLS) model interacting with Chern-Simons field under the influence of the quantum potential.

In the above mentioned interpretation of quantum mechanics the quantum particle moves under the action of a force which along with classical potential includes a contribution from the quantum potential. If instead of classical particle we consider Chern-Simons solitons, then subject to the influence of intensity  $s$  quantum potential, it could represent the stochastically quantized anyons. In this case we can expect quantization condition on the Chern-Simons coupling constant, an affect on the anyonic parameter and the appearance of the zero point fluctuation for the statistical flux.

## 2. CHERN-SIMONS SOLITONS HIERARCHY

We consider the Chern-Simons gauged Nonlinear Schrödinger model (the Jackiw-Pi model) with nonlinear quantum potential term of strength  $s$ :

$$L = \frac{\kappa}{2}\epsilon^{\mu\nu\lambda}A_\mu\partial_\nu A_\lambda + \frac{i}{2}(\bar{\psi}D_0\psi - \psi\bar{D}_0\bar{\psi}) - \bar{\mathbf{D}}\bar{\psi}\mathbf{D}\psi + s\nabla|\psi|\nabla|\psi| + g|\psi|^4, \quad (1)$$

where  $D_\mu = \partial_\mu + ieA_\mu$ , ( $\mu = 0, 1, 2$ ). Classical equations of motion are

$$iD_0\psi + \mathbf{D}^2\psi + 2g|\psi|^2\psi = s\frac{\Delta|\psi|}{|\psi|}\psi, \quad (2a)$$

$$\partial_1 A_2 - \partial_2 A_1 = \frac{e}{\kappa}\bar{\psi}\psi, \quad (2b)$$

$$\partial_0 A_j - \partial_j A_0 = -\frac{e}{\kappa}i\epsilon_{jk}(\bar{\psi}D_k\psi - \psi\bar{D}_k\bar{\psi}), \quad (j, k = 1, 2). \quad (2c)$$

As we mentioned in Introduction the extension of the linear Schrödinger equation with  $s \neq 0$  term has been considered before [11-14], and was linearized for *any*  $s$  [12]. But, when one adds the nonlinear self-interaction and the gauge field, integrability of the model is allowed only under some restrictions on the coupling parameters. For  $s < 1$ , and this

is only the case where we find the self-duality condition, we decompose  $\psi = e^{R-iS}$  and introduce new rescaled variables

$$\begin{aligned} t &= (1-s)^{1/2}\tilde{t}, \quad S = (1-s)^{1/2}\tilde{S}, \\ A_0 &= (1-s)\tilde{A}_0, \quad \mathbf{A} = (1-s)^{1/2}\tilde{\mathbf{A}}. \end{aligned} \quad (3)$$

Then for new function  $\tilde{\psi} = \exp(R - i\tilde{S})$ , we get the gauged NLS system (the Jackiw-Pi model)

$$i\tilde{D}_0\tilde{\psi} + \tilde{\mathbf{D}}^2\tilde{\psi} + \frac{2g}{1-s}|\tilde{\psi}|^2\tilde{\psi} = 0, \quad (4a)$$

$$\partial_1\tilde{A}_2 - \partial_2\tilde{A}_1 = \frac{e}{\kappa(1-s)^{1/2}}\tilde{\psi}\tilde{\psi}, \quad (4b)$$

$$\partial_{\tilde{0}}\tilde{A}_j - \partial_j\tilde{A}_0 = -\frac{e}{\kappa(1-s)^{1/2}}i\epsilon_{jk}(\tilde{\psi}\tilde{D}_k\tilde{\psi} - \tilde{\psi}\tilde{D}_k\tilde{\psi}), \quad (j, k = 1, 2), \quad (4c)$$

but with new coupling constants  $\tilde{\kappa} = \kappa(1-s)^{1/2}$ ,  $\tilde{g} = g/(1-s)$ . Since invariance of the density  $\rho = \bar{\psi}\psi = \tilde{\bar{\psi}}\tilde{\psi}$ , the Gauss law (4b) modifies the value of magnetic flux, which depends now of  $s$ . Only when the coupling constants are restricted by the condition

$$\frac{g\kappa}{e^2} = (1-s)^{\frac{1}{2}}, \quad (5)$$

static solutions of the system (4) with the gauge potential

$$\tilde{A}_0 = \frac{e}{\kappa(1-s)^{1/2}}\tilde{\bar{\psi}}\tilde{\psi},$$

require the self-dual Chern-Simons equations,

$$\tilde{D}_-\tilde{\psi} = 0, \quad (6a)$$

$$\partial_1\tilde{A}_2 - \partial_2\tilde{A}_1 = \frac{e}{\kappa(1-s)^{1/2}}\tilde{\bar{\psi}}\tilde{\psi}. \quad (6b)$$

Condition (5) extends for  $s \neq 0$ , the well-known Jackiw-Pi constraint. Namely, self-duality equations in the presence of quantum potential survive only if its strength is restricted by  $s = 1 - g^2\kappa^2/e^4$ .

To solve the system (6) we insert  $\mathbf{A}$  from the first equation to the second one and will get for the density  $\rho = \bar{\psi}\psi$  the Liouville equation

$$\Delta \ln \rho = -2\frac{e^2}{\kappa(1-s)^{1/2}}\rho. \quad (7)$$

The radially symmetric solutions, which have been constructed by G. W. Walker [17], and discussed in [3],

$$\rho(r) = 4\frac{\kappa(1-s)^{1/2}N^2}{e^2r^2}\left[\left(\frac{r}{r_0}\right)^N + \left(\frac{r_0}{r}\right)^N\right]^{-2}, \quad (8)$$

would be regular if  $N \geq 1$ . Then, like in [3], from regularity of the gauge potential  $\mathbf{A}$  we can fix the phase of  $\tilde{\psi} = \exp(R - i\tilde{S})$  as  $\tilde{S} = (N - 1)\theta$ ,  $\theta = \arctan x_2/x_1$ , and restrict  $N$  to be an integer for single-valued  $\tilde{\psi}$ . However, the auxiliary function  $\tilde{\psi}$  is not the physical one, this is why fixing an integer  $N$  is not sufficient for single valuedness of the original function  $\psi = \exp(R - i(1 - s)^{1/2}(1 - N)\theta)$ . It would be now that the phase is restricted as  $0 < S \leq 2\pi(1 - s)^{1/2}$ , and in general we have the problem with angular defect in the plane  $2\pi(1 - (1 - s)^{1/2})$ , describing a cone. It is easy to see to avoid this complication an integer valued must be the product

$$(1 - s)^{1/2}(N - 1) = n, \quad (9)$$

valuedness of which for any integer  $N$ , requires an integer valuedness of

$$(1 - s)^{1/2} = m, \quad (10)$$

and as the consequence of (5), we obtain the quantization condition

$$\frac{g\kappa^2}{e^2} = m, \quad (m = 1, 2, 3\ldots). \quad (11)$$

The last one means that in the presence of quartic self-interaction the Chern-Simons coupling constant and the quantum potential strength must be quantized

$$\kappa = m \frac{e^2}{g}, \quad s = 1 - m^2. \quad (12)$$

When  $m = 1$ , the quantum potential vanishes,  $s = 0$ , while the first constraint in (12) reduces to the Jackiw-Pi self-dual condition. It is worth to note that in our relation (12)  $g$  and  $\kappa$  coupling constants play the role similar to ones in the electro-magnetic duality condition derived by Dirac.

The self-dual system (6) for original  $(\psi, \mathbf{A})$  has the form, depending of the quantum potential with a coefficient proportional to the angular defect,

$$D_- \psi + ((1 - s)^{1/2} - 1) \frac{\partial_- |\psi|}{|\psi|} \psi = 0, \quad (13a)$$

$$\partial_1 A_2 - \partial_2 A_1 = \frac{e}{\kappa} \bar{\psi} \psi, \quad (13b)$$

and turn the energy

$$H = \int d^2 \mathbf{r} (\bar{\mathbf{D}} \bar{\psi} \mathbf{D} \psi - s \nabla |\psi| \nabla |\psi| - g |\psi|^4), \quad (14)$$

to vanish  $H = 0$ . It shows that under the influence of the quantum potential Chern-Simons solitons continue to be a zero energy configuration. Due to (9), and (10) the flux for solution (8) is quantized

$$\Phi = \int d^2 \mathbf{r} B = \frac{2\pi}{e} 2N(1 - s)^{1/2} = \frac{2\pi}{e} (2m)N, \quad (15)$$

where  $N = 1, 2, 3, \dots$  and value of  $m$  is fixed. Thus, the magnetic flux of our vortex/soliton is an even multiple  $m$  of the elementary flux quantum, which generalizes the Jackiw-Pi result related to the particular value  $m = 1$ . Here it is worth to note that the single-valuedness condition (9) may be satisfied also for particular real values for  $s$  and  $N$ . Thus, if  $(1 - s)^{1/2} = p/q$  is the rational number, then single-valuedness of  $\psi$  is allowed only for the specific sequence of integers  $N = 1, 1 + q, 1 + 2q, \dots$  or  $N = 1 + ql$ , where  $l = 0, 1, 2, \dots$ . In this case the flux is quantized as

$$\Phi = \frac{2\pi}{e}(2p)(l + \frac{1}{q}), \quad (16)$$

with  $1/q$  playing the role of the zero-point flux. For an irrational value  $(1 - s)^{1/2} = \alpha$  the flux is quantized

$$\Phi = \frac{2\pi}{e}2(n + \alpha), \quad (n = 0, 1, 2, \dots), \quad (17)$$

only for the irrational sequence  $N = 1 + n/\alpha$ .

The decomposition  $\psi = \exp(R - iS)$  is known in quantum mechanics as the Madelung fluid representation and has been explored for description of superconductivity [18]. In the problem (2) the velocity of associated Madelung fluid is definite as  $\mathbf{v} = -2(\nabla S - e\mathbf{A})$  and satisfies the conservation law

$$\partial_t \rho + \nabla \rho \mathbf{v} = 0. \quad (18)$$

For the self-dual flows, from Eq.(6a) we find the canonical Hamiltonian equations

$$\dot{x} \equiv v_1 = \frac{\partial \chi}{\partial y}, \quad \dot{y} \equiv v_2 = -\frac{\partial \chi}{\partial x}, \quad (19)$$

where the stream function  $\chi$ , playing the role of a Hamiltonian, has the form

$$\chi = (1 - s)^{1/2} \ln \rho. \quad (20)$$

For the soliton solution (8), function  $\rho$  is regular everywhere and has  $2(N - 1)$ -th order zero at the beginning of coordinates. This zero determines singularity of  $\chi$  and  $\mathbf{v}$ , such that near the singularity

$$\chi = 2(N - 1)(1 - s)^{1/2} \ln r \equiv -\frac{\beta}{2\pi} \ln r, \quad (21)$$

and the flow corresponds to the line vortex, the strength of which according to (9) must be quantized  $\beta = -4\pi n = 4\pi m(1 - N)$ . The velocity field near the center

$$\mathbf{v} = \frac{-2n}{r}(-\frac{y}{r}, \frac{x}{r}), \quad (22)$$

is the gradient of multivalued function  $\phi = -2n \arctan(y/x)$ , which fixes the value of the phase of the wave function  $\psi$ . As is well-known, functions  $\chi$  and  $\phi$  are conjugate harmonic functions, providing holomorphicity condition for the gauge potential near the

center. The above consideration shows that Chern-Simons soliton can be interpreted as a planar vortex in the Madelung fluid, having form of a linear vortex near the rotation point. This interpretation allows us to give the physical meaning for the vortex flow in the Madelung quantum liquid and answer the question posed in [19].

### 3. EXPONENTIALLY LOCALIZED CHERN-SIMONS SOLITONS

In previous section we considered self-duality reduction for  $s < 1$ . However, no solution for  $s > 1$  has been found. Now we show that in a special case of hyperbolic energy surface in the dynamics of Bloch electrons under the influence of Chern-Simons and quantum potentials, the problem admits an exact treatment also for  $s > 1$ . In the dynamics of Bloch electron in a solid a central role is played by the inverse effective mass tensor  $1/m \rightarrow \partial^2/\partial \mathbf{k}^2 E(\mathbf{k})$ , which is not necessary positive definite. When the Fermi surface is near the band boundary, the sign of the mass in orthogonal direction is negative. In this hyperbolic case, due to strong Bragg reflection from the boundary of the band, the electron propagates along trajectories which are parallel to the planes of the lattice, and  $E(\mathbf{k})$  has "saddle points", where the curvature of the surface may be positive in one direction and negative in another. For two-dimensional motion, in simplest case of the energy surface  $E(\mathbf{k}) = k_1^2/2m - k_2^2/2m$ , the effective mass matrix is  $M^{-1} = \text{diag}(1/m, -1/m)$  and the Bloch electron subject to the Lorentz force, imitated by Chern-Simons interaction, has a "dissipative" character [20], with the constant magnetic field playing the role of the damping. Then, in Lagrangian (1) and equations of motion (2) we replace the positive space metric  $\eta_{ij} = \text{diag}(1, 1)$  with the indefinite one  $\eta_{ij} = \text{diag}(1, -1)$  as follows

$$L = \frac{\kappa}{2} \epsilon^{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda + \frac{i}{2} (\bar{\psi} D_0 \psi - \psi \bar{D}_0 \bar{\psi}) - \bar{D}_a \bar{\psi} D^a \psi + s \partial_a |\psi| \partial^a |\psi| + g |\psi|^4, \quad (23)$$

$$i D_0 \psi + D_a D^a \psi + 2g |\psi|^2 \psi = s \frac{\partial_a \partial^a |\psi|}{|\psi|} \psi, \quad (24)$$

where  $\partial_a \partial^a = \eta^{ab} \partial_a \partial_b = \partial_1^2 - \partial_2^2$ . Now we rescale

$$\begin{aligned} t &= (s-1)^{1/2} \tilde{t}, \quad S = (s-1)^{1/2} \tilde{S}, \\ A_0 &= (s-1) \tilde{A}_0, \quad \mathbf{A} = (s-1)^{1/2} \tilde{\mathbf{A}}, \end{aligned} \quad (25)$$

and instead of  $\psi = \exp(R - iS)$ , introduce two real functions  $Q^\pm = \exp(R \pm \tilde{S})$ , satisfying the system

$$-\tilde{D}_0^- Q^+ + \tilde{D}_a^- \tilde{D}^{a-} Q^+ - \frac{2g}{s-1} Q^+ Q^- Q^+ = 0, \quad (26a)$$

$$\tilde{D}_0^+ Q^- + \tilde{D}_a^+ \tilde{D}^{a+} Q^- - \frac{2g}{s-1} Q^+ Q^- Q^- = 0, \quad (26b)$$

$$\partial_1 \tilde{A}_2 - \partial_2 \tilde{A}_1 = \frac{e}{\kappa(s-1)^{1/2}} Q^+ Q^-, \quad (26c)$$

$$\partial_0 \tilde{A}_j - \partial_j \tilde{A}_0 = \frac{e}{\kappa(s-1)^{1/2}} \epsilon_{jk} \eta_{kk} (Q^+ \tilde{D}_k^+ Q^- - Q^- \tilde{D}_k^- Q^+), \quad (j, k = 1, 2), \quad (26d)$$

where  $D_\mu^\pm = \partial_\mu \pm eA_\mu$ . This system represents the gauged version of the pair time-reversal invariant diffusion equations with the reaction term proportional to  $Q^+ Q^-$ . It is worth to note that switching on the Chern-Simons term leads to switching off the time-reversal invariance without changing the geometry of the classical trajectories. Moreover, instead of the local phase transformations for the original model (23), in the system (26) the Chern-Simons gauge field corresponds to the local rescaling of  $Q^\pm$ . For static configurations, when  $\tilde{A}_0 = -(e/\kappa(s-1)^{1/2})Q^+ Q^-$ , and

$$(s-1)^{1/2} = -\frac{g\kappa}{e^2}, \quad (27)$$

we have the self-duality equations

$$\tilde{D}_-^- Q^+ = 0, \quad \tilde{D}_+^+ Q^- = 0, \quad (28a)$$

$$[\tilde{D}_+^+, \tilde{D}_-^-] = -\frac{2e^2}{\kappa(s-1)^{1/2}} Q^+ Q^- = -[\tilde{D}_+^-, \tilde{D}_-^+], \quad (28b)$$

with  $D_\pm^a = D_1^a \pm D_2^a$ . Expressing  $\tilde{A}_\pm = \mp(1/e)\partial_\pm \ln Q^\mp$  and substituting to (28b) for  $\rho = Q^+ Q^-$  we obtain the Liouville equation

$$(\partial_1^2 - \partial_2^2) \ln \rho = 2 \frac{e^2}{\kappa(s-1)^{1/2}} \rho. \quad (29)$$

As is well-known this equation admits the general solution

$$\rho = \frac{8}{\alpha_0} \frac{A'(x_+) B'(x_-)}{[A(x_+) + B(x_-)]^2}, \quad (30)$$

written in terms of two arbitrary functions  $A(x_+)$  and  $B(x_-)$  of  $x_+ = x_1 + x_2$  and  $x_- = x_1 - x_2$  respectively, where  $\alpha_0 = 2e^2/\kappa(s-1)^{1/2}$ . It has been considered before as 1+1 dimensional evolution equation with  $x$  coordinate considered as a time variable. However, the known regular soliton solutions [21,22], decay in all directions except the soliton world line, which leads to the divergent integral for  $\rho$ , this is why they have no physical meaning in our problem. Instead of it we choose  $A$  and  $B$  functions in the form

$$A = (a+1) \coth^{2k+1} \frac{\alpha}{2} x_+, \quad B = (a-1) \tanh^{2l+1} \frac{\beta}{2} x_-, \quad (31)$$

then we get

$$\rho = \frac{2^{1-2(k+l)}(1-a^2)(2k+1)(2l+1)\alpha\beta \sinh^{2k} \alpha x_+ \sinh^{2l} \beta x_-}{\alpha_0[(a+1) \cosh^{2k+1} \frac{\alpha}{2} x_+ \cosh^{2l+1} \frac{\beta}{2} x_- + (a-1) \sinh^{2k+1} \frac{\alpha}{2} x_+ \sinh^{2l+1} \frac{\beta}{2} x_-]^2}.$$

For parameter  $a > 0$  this solution is nonsingular in the whole  $(x_1, x_2)$  plane. Sign definiteness of  $\rho$  requires  $k, l$  to be integer, while regularity at the "light-cone",  $x_+ = 0$  or  $x_- = 0$  is valid for  $k \geq 0, l \geq 0$ . In contrast to the algebraic Chern-Simons solitons (8), solution (31) is exponentially decreasing in the plane for the "future" and "past" null infinities as  $e^{-\alpha|x|}$  or  $e^{-\beta|x|}$ , while for the "time-like" and "space-like" infinities as  $e^{-(\alpha+\beta)|x|}$  (here without loose of generality we put  $\alpha > 0, \beta > 0$ ). It is worth to note that our solutions are related with exponentially localized planar solitons ( $EL_2$ ) for Davey-Stewartson-I (DS-I) equation [23] (also known as dromions [24]). These solutions have a rich integrable dynamics [25], and are reducible to the Liouville equation for  $|\psi|$ , and to the d'Alembert equation  $\partial_+ \partial_- \arg \psi = 0$  for  $\arg \psi$ , like in the DS-II case [26]. This reduction provides an integrable dynamics for our exponentially localized Chern-Simons solitons [27]. Moreover, comparing to (8), having for  $N > 1$  the zero at the beginning of coordinates, the zeroes for solution (31)  $\rho \sim x_+^{2k} x_-^{2l}$  are located along the "light cone"  $x_+ = 0, x_- = 0$  for  $k > 0$  and  $l > 0$  correspondingly. For  $k = 0$  and  $l = 0$ , we have exponentially localized soliton located at the beginning of coordinates, without zeroes and the rotation invariance at this point (Fig.1). If only one of the  $k, l$  values is vanishing, then we have two soliton solution represented on Fig.2. and symmetrical under one of the light cone directions. When both numbers do not vanish we get four-soliton solution, represented on Fig.3, with zeroes along the whole light cone. For  $k > 1, l > 1$ , number of solitons remains four, however the order of zeroes on the light cone is changing. As is well known, zeroes of  $\rho$ , which produce singularities for  $\ln \rho$ , lead to singularities of the gauge potential [3]. In the case (8) these singularities, corresponding to the Aharonov-Bohm type potential, can be removed by fixing the phase of the wave function at the singular point. So the phase becomes the angle variable on the plane with finite range of values. For solution (31), the singularities of potential  $\mathbf{A}$  are located along the light cone and can be compensated by derivation of function  $\tilde{S}$  as

$$\partial_+ \tilde{S} = 2k \frac{1}{x_+}, \quad \partial_- \tilde{S} = -2l \frac{1}{x_-}, \quad (32)$$

or

$$\tilde{S} = \frac{1}{2} \ln \frac{x_+^{2k}}{x_-^{2l}}. \quad (33)$$

This allows us to define exact solutions of Eqs. (26) and original system (24)

$$\psi = \sqrt{\rho} \left( \frac{x_-^l}{x_+^k} \right)^{i\sqrt{s-1}}. \quad (34)$$

For particular values  $k = l$ , the phase  $\tilde{S} = k \ln \left| \frac{x_+}{x_-} \right|$  defines the hyperbolic analog of the singular Aharonov-Bohm potential

$$a_i = \frac{2k}{e} \epsilon_{ij} \frac{x_j}{x_1^2 - x_2^2} = \frac{k}{e} \epsilon_{ij} \eta_{jj} \partial_j \ln r^2, \quad (35)$$

where  $r^2 = x_1^2 - x_2^2$  in the "time-like" quadrants II ( $x_1 > 0$ ) and IV ( $x_1 < 0$ ) of the plane, while  $r^2 = x_2^2 - x_1^2$  in the "space-like" quadrants I ( $x_2 > 0$ ) and III ( $x_2 < 0$ ), parameterized



as  $x_2 = \pm r \cosh \theta$ ,  $x_1 = \pm r \sinh \theta$  in I and III, and  $x_1 = \pm r \cosh \theta$ ,  $x_2 = \pm r \sinh \theta$  in II and IV, correspondingly. Here  $0 < r < \infty$ ,  $-\infty < \theta < +\infty$ . The potential (35) has singularities not only at the beginning of coordinates, as the Aharonov-Bohm potential, but also along the "light-cone",  $x_+ = 0$ ,  $x_- = 0$  and may be presented as a singular pure gauge

$$a_i = -\frac{1}{e} \partial_i \tilde{S} = -\frac{2k}{e} \partial_i \theta, \quad (36)$$

where,  $\tilde{S} = 2k\theta$ , and

$$\theta = \begin{cases} \tanh^{-1} \frac{x_1}{x_2} & \text{in I and III,} \\ \tanh^{-1} \frac{x_2}{x_1} & \text{in II and IV.} \end{cases} \quad (37)$$

The magnetic flux associated with soliton solutions (23) has the form

$$\int \tilde{B} d^2x = \frac{e}{\kappa(s-1)^{1/2}} \int \rho d^2x = \frac{2}{e} \ln a, \quad (38)$$

independent of  $k$  and  $l$ , and is not quantized. Moreover, the phase of  $\psi$  includes the hyperbolic rotation angle  $\theta$  which is valued on the whole real line. So no restriction of single-valuedness arises, and continual parameters  $s$  and  $\kappa$  must be restricted only by the relation (27).

At the end of this section we represent another problem related to our self-dual system (28). Namely, the hyperbolic self-duality equations (28) are equivalent to  $SO(2,1)/O(1,1)$  self-dual  $\sigma$  model:

$$\partial_1 \mathbf{s} - \mathbf{s} \wedge \partial_2 \mathbf{s} = 0, \quad (39)$$

written in the tangent space representation for moving frame

$$D_\mu^\mp \mathbf{n}_\pm = \mp 2 \sqrt{\frac{\alpha_0}{2}} Q_\mu^\pm \mathbf{s}, \quad (40a)$$

$$\partial_\mu \mathbf{s} = (-\sqrt{\frac{\alpha_0}{2}})(Q_\mu^+ \mathbf{n}_- - Q_\mu^- \mathbf{n}_+), \quad (40b)$$

to the one sheet hyperboloid  $\mathbf{s}^2 = -s_1^2 + s_2^2 - s_3^2 = -1$  as the constraints  $Q_-^+ = 0$ ,  $Q_+^- = 0$ , with the following identification  $Q_+^+ \equiv Q^+$ ,  $Q_-^- \equiv Q^-$ . In terms of the stereographic projection of the hyperboloid

$$S_\pm = \frac{2\xi_\pm}{1 + \xi_+ \xi_-}, \quad S_3 = \frac{1 - \xi_+ \xi_-}{1 + \xi_+ \xi_-}, \quad (41)$$

Eqs. (39) are just the chirality conditions

$$\partial_+ \xi_+ = 0, \partial_- \xi_- = 0, \quad (42)$$

having the general solution  $\xi_+ = \xi_+(x_-)$ ,  $\xi_- = \xi_-(x_+)$ . These functions correspond to the general solution of the Liouville equation (30) with identification  $\xi_- = A^{-1}(x_+)$ ,  $\xi_+ = B(x_-)$ . Then, our vortex configurations (31) generate solution of (39),(42) with

$$\xi_- = (a+1)^{-1} \tanh^{2k+1} \frac{\alpha}{2} x_+, \quad \xi_+ = (a-1) \tanh^{2l+1} \frac{\beta}{2} x_-. \quad (43)$$

which for  $a > 0$  are regular everywhere on the plane with  $S_3 > 0$ . The last condition means that the solutions for  $a > 0$  are non-topological. To have topologically nontrivial configurations we need to consider  $a < 0$  case, which would produce singularity for  $\rho$ . For the regular solution  $\mathbf{s}$  with  $k > 0$  or (and)  $l > 0$  the plateau along the light cone appears, leading to the zeroes for  $\rho$  in (31).

#### 4. CONCLUSIONS

In conclusions we stress that two long-range interactions considered in the present paper, the Chern-Simons gauge interaction introduced to physics by S. Deser, R. Jackiw and S. Templeton [1] and the quantum potential introduced by L. de Broglie [6] and D. Bohm [7], are compatible in supporting static soliton solutions in 2+1 dimensions, with arbitrary  $N$ , only when the coupling constants for both interactions are quantized.

A further remark concerns the special case  $s = 1$ , when the Madelung hydrodynamical formulation of quantum mechanics becomes the Euler equation and the continuity one [15]. For a special form of the nonlinearity an additional higher symmetry of the equations, related to the membrane theory, has been described recently [30]. It would be interesting to extend this work to include a Chern-Simons gauge field, in attempt to produce completely integrable model with infinite number of conservation laws.

Finally, we note existence of another integrable reduction of the models (1) and (23), when the fields are independent on one of the space directions. In this case for  $s < 1$  the model reduces to the BF gauged NLS, equivalent to NLS [28], and for  $s > 1$ , to BF gauged reaction-diffusion analog of NLS, equivalent to the reaction-diffusion system. The last one appears in the Jackiw-Teitelboim gravity and admits dissipative analog of solitons, called *dissipatons* and related to the black holes of the model [29]. The interaction of dissipatons show the resonance character [16].

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## Figures

**Fig. 1.** Contour and 3D plot of one soliton solution ( $k = 0$ ,  $l = 0$ ) for  $a = 0.7$  in  $(x,y)$  plane.

**Fig. 2.** Contour and 3D plot of two soliton solution ( $k = 1$ ,  $l = 0$ ) for  $a = 0.7$  in  $(x,y)$  plane.

**Fig. 3.** Contour and 3D plot of four soliton solution ( $k = 1$ ,  $l = 1$ ) for  $a = 0.7$  in  $(x,y)$  plane.